

On the nonlinear generalization of the Fock method

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1995 J. Phys. A: Math. Gen. 28 4973

(<http://iopscience.iop.org/0305-4470/28/17/026>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.68

The article was downloaded on 02/06/2010 at 01:00

Please note that [terms and conditions apply](#).

On the nonlinear generalization of the Fock method

V M Eleonsky† and V G Korolev‡

Lukin Research Institute of Physical Problems, Zelenograd, Moscow, 103460, Russia

Received 10 October 1994, in final form 20 February 1995

Abstract. The generalized Fock method is presented in the case of nonlinear dependence of the number-of-quanta operator on the Hamiltonian operator. That generalization is used for an analysis of the inverse problem in the case of strictly discrete spectra in \mathbb{R}^1 . The relationship between the quantum description on a basis of the Fock approach and the classical limit of this description is analysed.

1. Introduction

Interest in analysis of the Schrödinger spectral problem and in the search for new classes of ‘integrable’ potentials has quickened considerably in recent years [1, 2].

To investigate this problem several methods have been used: different advanced versions of the Darboux (factorization) method [3–6] and related approaches, for example, a supersymmetry analysis [7–9], an analysis of higher symmetries of the Schrödinger equation [10–12], a construction of associated nonlinear algebras [13] and, finally, an analysis of q -deformed oscillators [14–18].

These investigations have a profound impact on the development of the spectral theory of the ‘main’ operator in fundamental quantum physics—the Schrödinger operator [19–21]—and may serve as a basis for an advancement of approximate methods for solving a number of problems of substantial interest for applications. Note the problem on reconstruction of regular potentials with a prescribed strictly point-energy spectrum (for example, with a spectrum containing a given set of gaps on the background of a strictly equidistant sequence of levels). This problem arises in the development of semiconductor devices based on so-called ‘nanostructures’ (in particular, the quantum wells [22, 23]). Here an analysis of bifurcations of creation and annihilation of energy gaps under changes of potential parameters is of certain interest (from the standpoint of widening the scope of the working of those devices).

On the other hand, the methods mentioned above lead to a natural generalization of the concept of Fock’s creation and annihilation operators (the operators shifting along the spectrum) to the case of nonlinear oscillators. This branch attracts particular attention in relation to the study of coherent (or squeezed) states in nonlinear quantum optics [24].

The basis for our approach is a development of the well-known Fock method in the theory of harmonic oscillators [25]. This method essentially uses the introduction of the number-of-quanta operator (hereafter called ‘the number operator’) N which depends

† E-mail address: eleon@nonlin.msk.ru

‡ E-mail address: korolev@nonlin.msk.ru

linearly on the Hamiltonian operator H . Our approach is based on the generalization of the Fock method to the case of the nonlinear form of this dependence.

The first version of this approach was presented in [26, 27], where it was used mainly for an analysis of the problem on reconstruction of potentials (regular in \mathbb{R}^1 and growing at infinities) with prescribed size and location of the unique energy gap on the background of the strictly equidistant spectrum. There the number operator N was a polynomial of the third degree of H .

As mentioned in [27] and then studied in detail in our papers [28, 29], a consequence of this approach allows us to obtain potentials with more complicated structure in comparison to the class of anharmonic oscillators. In particular, we found the class of potentials with the asymptotics

$$U(x)_{x \rightarrow \pm\infty} \sim \frac{1}{2} \left(\frac{\omega x}{3} \right)^2 + \text{constant } |x| \sin \left(\frac{\omega}{\sqrt{3}} x^2 + \alpha \right)$$

having a spectrum which is a strictly equidistant continuation of a triplet of 'ground' states (in terms of the operator N), shifted arbitrarily relative to each other.

The latter class was obtained and studied in the remarkable paper [30] by Veselov and Shabat in the framework of an advanced factorization method (the 'dressing chains' method). It was shown that such a type of spectrum (a combination of M arithmetic series) is valid for the shift operator of arbitrary degree M of the momenta operator. In [30] it was also mentioned that a physical application of such potentials has recently arisen in string theory.

An advanced nonlinear generalization of the Fock approach allows the analysis of potentials leading to substantially non-equidistant (for example, quadratic) spectra. Another spectrum type that may be analysed using this method is a discrete spectrum with a finite number of levels.

All the above leads us to the necessity of a presentation of the nonlinear generalization of the Fock approach for a more general form of nonlinear dependence of the operator N on H .

Furthermore, one of the important advantages of the approach we propose is the possibility of studying both a quantum dynamical system and its classical limit simultaneously. In this paper we attempt to expose the nonlinear generalization of the Fock approach for both quantum and classical cases in a parallel way.

The main results of this paper can be formulated as follows.

(i) It is shown that the specification of a functional dependence of the number operator on the Hamiltonian operator for the Schrödinger problem in \mathbb{R}^1 leads to a one-dimensional mapping of its eigenelements. That mapping determines the structure of the energy spectrum.

(ii) The conditions of solvability of equations for generalized creation and annihilation operators (the operators of shift along a spectrum) in the case of the natural Hamiltonian operator ($H = \frac{1}{2} p^2 + U(x)$, $x \in \mathbb{R}^1$) lead to a nonlinear differential equation for potentials $U(x)$, which realize the spectrum of H , determined by the one-dimensional mappings mentioned above.

(iii) Analysis of the relationship between N and H under changes of the structure parameters makes it possible to investigate bifurcations of creation and annihilation of energy gaps in the spectrum of H and corresponding bifurcations of potential relief.

(iv) In the classical limit the method we present is directly related to the well-known inverse problem of classical mechanics—the problem on reconstruction of a potential using a given dependence of oscillation period on energy. In particular, the equation for a quantum

potential with quasi-equidistant† spectrum in the classical limit ($\hbar \rightarrow 0$) reduces to the equation for a classical potential. Solving it provides a potential which admits isochronous motion in a certain range of parameter values.

(v) Among all the possible branches of the eigenelements mapping, the unique branch survives in the classical limit and, thus, the possibility of existence of multivalued mappings disappears by transition from q - to c -numbers.

The approach described in this paper admits further generalizations related to an analysis of the inverse problem for substantially non-equidistant discrete spectra and to the case of several degrees of freedom.

The paper is arranged as follows. In section 2 we expose the generalization of the Fock approach to a sufficiently arbitrary form of nonlinear relation between the operators N and H . In section 3 this approach is illustrated by the analysis of polynomial relationships of these operators. In section 4 the classical limit of this method is considered. Finally, in section 5 this classical version of our approach is illustrated, as in the quantum case, by the polynomial dependence of the function $N(H)$ on the Hamiltonian function.

2. Nonlinear generalization of the Fock approach: the quantum case

Let H be a Hamiltonian operator of the quantum dynamical system which has a structure that will be defined below. Analogues of the creation and annihilation operators L and L^\dagger are defined by the relations

$$LL^\dagger = N(H) \quad L^\dagger L = \tilde{N}(H) \quad (2.1)$$

or, equivalently,

$$LL^\dagger = N(H) \quad [L, L^\dagger] = \Delta(H) \quad \Delta(H) \equiv \tilde{N} - N. \quad (2.2)$$

Here N, \tilde{N} are the given functions of the Hamiltonian operator. Moreover, the non-negative operator $N(H)$ can be treated as an analogue of the number operator [25].

Then we find that

$$[N, L] = L\Delta(H) \quad [N, L^\dagger] = -\Delta(H)L^\dagger. \quad (2.3)$$

A condition of commutation $[N, H] = 0$ is satisfied if the operators L, L^\dagger obey the equations

$$[H, L] = L\Omega(H) \quad [H, L^\dagger] = -\Omega(H)L^\dagger \quad (2.4)$$

where $\Omega(H)$ is a Hermitian operator depending on the Hamiltonian operator H . Taking into account equations (2.4), equation (2.3) leads to the relation

$$N(H + \Omega(H)) - N(H) = \Delta(H) \quad (2.5)$$

(see appendix A). By the given dependence of the number operator N on H this relation determines a dependence of the characteristic operator $\Omega(H)$ on H and on the structure parameters of the system. On the other hand, by the given characteristic operator $\Omega(H)$ and the function $\Delta(H)$, relation (2.5) can be considered as the functional equation relative to the operator $N(H)$.

Let (E, ν, ψ) be an eigenelement of the Schrödinger problem

$$H\psi = E\psi \quad N\psi = \nu\psi \quad \nu \geq 0. \quad (2.6)$$

† i.e. either strictly equidistant or equidistant except for a finite number of arbitrary gaps.

Then by virtue of relation (2.3) the function $\psi' \equiv L\psi$ obeys the equation

$$N\psi' = (\nu(E) + \Delta(E))\psi' \equiv \nu'\psi'. \quad (2.7)$$

Thus, if $L\psi \neq 0$ then ψ' is an eigenfunction of the operator N corresponding to the eigenvalue $\nu' = \nu(E) + \Delta(E)$. Moreover, (E', ν', ψ') is an eigenelement of the Schrödinger problem (2.6) with the eigenvalue E' determined by the relation

$$\nu' \equiv \nu(E') = \nu(E) + \Delta(E) \geq 0. \quad (2.8)$$

Thus, the action of the operator L on the eigenelements of the Schrödinger problem (2.6) generates the one-dimensional mapping $(\nu \Rightarrow \nu', E \Rightarrow E', \psi \Rightarrow \psi' \equiv L\psi)$, if problem (2.6) is solvable for at least one eigenelement (E, ν, ψ) . This mapping is to be stopped at the n th step if at the $(n+1)$ th step $\nu_{n+1} < 0$.

We recall that in the case when N depends linearly on H the ground state of the quantum system is defined by the solution of the Schrödinger equation (2.6) corresponding to the zero eigenvalue $\nu(E) = 0$ of the number operator N . Let us keep this definition and note that, generally, the problem

$$H\psi_0 = E_0\psi_0 \quad N\psi_0 = 0 \quad \nu(E_0) = 0 \quad (2.9)$$

can be solved for several real roots of the equation $\nu(E_0) = 0$. This situation is realized, for example, in the case when $N(H)$ is a polynomial of degree $M \geq 3$ in H .

In other words, the problem on the ground state (2.9) can be degenerated relative to the zero eigenvalue of the operator N . Each of the roots of the equation $\nu(E_0) = 0$ is the ground state for its own subsequence of the eigenvalues of the operator H generated by the shift operator L . Each of these subsequences can contain either a finite or an infinite number of elements. Combinations of these subsequences determine the possible spectrum of the operator H .

Furthermore, the sequence function associated with the L -mapping and given in implicit form by the relation (2.8) can be multivalued at some values of the structure parameters.

It is directly related to the fact that relation (2.5) in the general case determines several characteristic operators $\Omega(H)$ and, as a consequence, a corresponding number of roots of equation (2.4). However, only one characteristic operator (and corresponding branch of the multivalued sequence function) does not disappear by going to the classical limit ($\hbar \rightarrow 0$). In reality, as will be shown later, the classical analogue of relation (2.5) determines a unique characteristic function $\Omega(H)$, where H is a classical Hamiltonian function.

This important feature of the generalized Fock approach is a consequence of a number of facts. By the given functions (N, Δ) and the corresponding spectrum of H , the solutions of equation (2.4) determine pairs of operators (H, L) for different branches of multivalued (in the general case) mapping. For the branch that has the classical analogue the problem can be solved for a scalar wavefunction and for the Hamiltonian operator of the natural form $(H = \frac{1}{2}P^2 + U)$. For other branches the condition of solvability in \mathbb{R}^1 can be realized for wavefunctions and Hamiltonian operators of a more complicated nature. In particular, by the values of structure parameters leading to multivaluedness of the one-dimensional mapping of the eigenelements, the corresponding Schrödinger problem in \mathbb{R}^1 becomes solvable when we introduce pseudo-spin degrees of freedom.

3. The case of polynomial relations between the number operator and the Hamiltonian operator

As an example confirming the statements presented in the previous section, let us consider the case when the operators N, \tilde{N} depend on the Hamiltonian operator H nonlinearly as

polynomials of the third degree:

$$N(H) = \sum_{m=0}^3 N_m H^m \quad \tilde{N}(H) = \sum_{m=0}^3 \tilde{N}_m H^m. \tag{3.1}$$

Here (N_m, \tilde{N}_m) are the structure parameters of the system. In this case equation (2.5), which determines the characteristic operator $\Omega(H)$, takes the form

$$N_3 \Omega^3 + (N_2 + 3N_3 H) \Omega^2 + (N_1 + 2N_2 H + 3N_3 H^2) \Omega = \Delta_0 + \Delta_1 H + \Delta_2 H^2 + \Delta_3 H^3 \tag{3.2}$$

where $\Delta_j \equiv \tilde{N}_j - N_j$. Depending on the values of the structure parameters, equation (3.2) specifies no more than three real characteristic operators. Let these values be such that $\Omega_1, \Omega_2, \Omega_3$ are solutions of equation (3.2). Then the equations

$$[H, L] = L \Omega_j(H) \quad j = 1, 2, 3 \tag{3.3}$$

determine, in the general case, three (H, L) -pairs of operators and three branches of L -mapping:

$$L^{(j)} : \psi_n^{(j)} \rightarrow L^{(j)} \psi_n^{(j)} \quad E_n^{(j)} \rightarrow E_{n+1}^{(j)} = E_n^{(j)} + \Omega_j(E_n) \tag{3.4}$$

$$\nu(E_n^{(j)}) > 0 \quad \nu(E_{n+1}^{(j)}) \geq 0 \quad j = 1, 2, 3.$$

Here

$$\nu(E) \equiv \sum_{m=0}^3 N_m E^m. \tag{3.5}$$

Remark. Note that by the given dependences $N(H), \tilde{N}(H)$ one can write an equation for determining the (H, L) -pair that has a more general structure in comparison with (3.3), if one requires the commutation condition $[H, N] = 0$. For example, in the case when the operators N, \tilde{N} are polynomials of the third degree of H , this equation has the form

$$3N_3 K^1 H^2 + (2N_2 K^1 + 3N_3 K^2) H + (N_1 K^1 + N_2 K^2 + N_3 K^3) = L(\Delta_0 + \Delta_1 H + \Delta_2 H^2 + \Delta_3 H^3) \tag{3.6}$$

where K^i are recurrent commutators: $K^0 \equiv [H, L], K^{i+1} = [H, K^i], i \geq 0$. However, we are not aware of any methods for solving this equation, apart from the use of relation (3.3) that reduces (3.6) to (3.2), which makes this generalization a formal one.

Let us consider the case that one of the characteristic operators (for example, Ω_1) does not depend on H . In this case the equation

$$[H, L] = \omega L \quad \omega \equiv \Omega_1 \equiv \text{constant} \tag{3.7}$$

determines the (H, L) -pair of operators and the L -mapping on the equidistant part(s) of the spectrum of H (the same equation was obtained by the 'dressing chains' method in [30]).

The corresponding sequence function has the form

$$E_{n+1} = E_n + \omega \quad \nu(E_n) > 0 \quad \nu(E_{n+1}) \geq 0. \tag{3.8}$$

Such a situation is realized when the structure parameters obey the relations

$$\begin{aligned} \Delta_0 &= N_1 \omega + N_2 \omega^2 + N_3 \omega^3 & \Delta_1 &= 2N_2 \omega + 3N_3 \omega^2 \\ \Delta_2 &= 3N_3 \omega & \Delta_3 &= 0. \end{aligned} \tag{3.9}$$

If the ground state defined by problem (2.9) is non-degenerate in terms of the operator N (i.e. the structure parameters are such that the equation $\nu(E_0) = 0$ possesses a unique real root), then the sequence function (3.8) determines a strictly equidistant spectrum of H .

If the ground state is degenerate in the operator H and $\{E_0^{(1)}, E_0^{(2)}, E_0^{(3)}\}$ is a triplet of eigenvalues of H corresponding to the zero eigenvalue of N , then the sequence function (3.8) defines the following two types of general structure of the spectrum H .

(i) The spectrum of H consists of two equidistant groups of levels. The lower group is built by the L -mapping of the lowest eigenvalue of H among 'the ground triplet' $E_0^{(i)}$; the mapping is stopped at the n th step due to the condition $\nu_{n+1} = 0$. Thus, the lower group contains a finite number of levels. The top, unbounded above, countable group of levels is separated from the lower one by a gap. The size of the gap in the general case is incommensurable with the step (equal to ω) in the equidistant parts of the spectrum. The location of the gap in the equidistant spectrum is determined by values of the structure parameters of the system.

(ii) The spectrum of H is an equidistant continuation (with step ω) of the triplet of ground states $\{E_0^{(1)}, E_0^{(2)}, E_0^{(3)}\}$ (this case was also studied in [30]).

The following choice of the structure parameters corresponds to the case considered in [26] and other papers:

$$\begin{aligned} N_0 &= -\omega(C_1 + \omega^2) - C_2 & N_1 &= 2(C_1 + 3\omega^2) \\ N_2 &= -12\omega & N_3 &= 8. \end{aligned}$$

Here (C_1, C_2) are the values of the two integrals [27]

$$C_1 = x \left\{ -\frac{1}{2} V_{xxx} + 3(V^2)_x + 2(\omega x)^2 V_x \right\} + \frac{1}{2} V_{xx} - 3V^2 \quad (3.10)$$

$$C_2 = -\frac{1}{4(\omega x)^2} \left\{ -\frac{1}{2} V_{xx} + 3V^2 + C_1 \right\}^2 + \frac{1}{4} (V_x)^2 - V^3 - C_1 V \quad (3.11)$$

of the equation for the potential $V \equiv U(x) - (\omega x)^2/2$:

$$\frac{1}{4} V_{xxxx} - \frac{3}{2} (V^2)_{xx} - (\omega x)^2 V_{xx} - 3\omega^2 x V_x = 0. \quad (3.12)$$

Expression (3.11) may also be considered as a differential equation of the second degree with two structure parameters. Note another convenient form of equation (3.11) as a system of two non-autonomous equations of the first degree is

$$\frac{1}{4} (V_x)^2 = \omega^2 (xV - W)^2 + V^3 + C_1 V + C_2 \quad W_x = V. \quad (3.13)$$

Equation (3.12) was derived in our previous papers [26, 27] as a condition for solvability of the overdetermined system of linear equations for coefficients of the polynomial

$$L(p, x) = \sum_{m=0}^3 L_m(x) p^m. \quad (3.14)$$

(This system arises by solving (3.7) if one uses the form (3.14) for the shift operator L .)

A direct analogue of that equation was found in [11] in the theory of higher symmetries of the Schrödinger equation. In the theory of 'dressing chains' a relationship of such equations with the Painlevé transcendents was established by Veselov and Shabat [30].

In [26, 27] we presented some results on the structure of the spectrum of the Schrödinger problem for some explicit solutions of equation (3.12) which were found using the Darboux method at certain values of the structure parameters.

The method proposed in this paper makes it possible to determine completely the structure of the spectrum of the Hamiltonian operator associated with solutions of the equation (3.12) regular in \mathbb{R}^1 , and to investigate the bifurcations of the creation (annihilation)

of a gap on the background of an equidistant spectrum. Expressions (3.8) at $C_1 > 0$, taking into account (3.5), lead to the sequence function

$$\begin{aligned} E_{n+1} &= E_n + \omega \\ \nu(E_n) &= (2E_n - \omega)^3 + C_1(2E_n - \omega) - C_2 > 0. \end{aligned} \tag{3.15}$$

Here the ground state $E_0(C_1, C_2)$ is unique, and the L -mapping is definite (one-valued) and determines the unbounded sequence of energy eigenvalues given above. The spectrum of H is a strictly equidistant one with distance between levels equal to ω .

If $C_1 < 0$ then the condition $\nu(E) = 0$ determines, in the general case, the triplet of ground states $\{E_0^{(1)}, E_0^{(2)}, E_0^{(3)}\}$ and in addition the sequence function is multivalued, i.e. at $E_{cr}^{\pm} \equiv -\omega/2 \pm \sqrt{-C_1/3}$ on the equidistant branch (3.15) a pair of new branches arises:

$$E_{n+1} = -\frac{1}{2}E_n + \frac{\omega}{4} \pm \frac{1}{2}\sqrt{-C_1 - 3\left(E_n + \frac{\omega}{2}\right)^2}. \tag{3.16}$$

Multivaluedness is excluded if L obeys equation (3.7).

The structure of the spectrum of H depends on the parameters C_1, C_2 as follows [28]: on the half-plane ($C_1 < 0, C_2$), a countable set of bounded regions exists where the spectrum is an equidistant continuation (with step ω) of the triplet of ground states. These regions are bounded by the ordered set of curves where an equidistant spectrum with gap is realized. The location of the gap (i.e. the number of levels in the lower equidistant part) is equal to the number of the curve. These curves can be found explicitly from the condition needed for the eigenvalue of N to be zero at the n th step of the L -mapping. Namely, the n th mapping of the initial point $E_0^{(1)} = E_0^{(1)}(C_1, C_2)$ leads to $\nu(E_n) = 0$ on the family of curves:

$$27 C_2^2 = -4[C_1 + 4(\omega n)^2]^2[C_1 + (\omega n)^2] \quad n = 0, 1, 2, \dots \tag{3.17}$$

Here the energy of the ground state is

$$E_0^{(1)} = \frac{1}{4} \frac{3C_2}{C_1 + 4(\omega n)^2} - \frac{\omega}{2}(n - 1) \quad n = 0, 1, 2, \dots \tag{3.18}$$

Numerical simulation, as well as analysis on the basis of Darboux transformations, showed that such a structure of the spectrum is realized on symmetric potentials regular in \mathbb{R}^1 with asymptotics $\lim_{x \rightarrow \pm\infty} \{U(x) - (\omega x)^2/2\} = 0$ and a finite number of local minima of the function $V(x)$ (their number linearly depends on the number of the curve (3.17)). Note that in the case of symmetric potentials it is convenient to go from the parameters C_1 and C_2 to the 'natural' parameters of the potential $U_0 \equiv U(0)$ and $U_2 \equiv U_{xx}(0)$ using the relations

$$C_1 = \frac{1}{2}[U_2 - \omega^2] - 3U_0^2 \quad C_2 = -U_0^3 - C_1U_0. \tag{3.19}$$

Then, for the case of symmetric potentials, expression (3.17) gives, apart from the line of harmonic oscillator ($n = 0$), two alternating sets of half-lines and half-parabolas in the plane $\{U_0, U_2\}$, corresponding to the families of potentials that provide equidistant spectra with gap [28]:

$$U_2^{(1)} = -12\omega n U_0 - \omega^2(8n^2 - 1) \quad U_0 > -\frac{2}{3}\omega(n + 1) \quad n = 2, 4, \dots \tag{3.20}$$

$$U_2^{(2)} = \frac{9}{2}U_0^2 - \omega^2(2n^2 - 1) \quad U_0 < -\frac{2}{3}\omega(n - 2) \quad n = 1, 3, \dots \tag{3.21}$$

The lower equidistant part of the spectrum contains n levels and the energy of the ground state (3.18) is written as

$$E_0^{(1)} = -\frac{1}{2}U_0 - \frac{1}{2}\omega(2n - 1) \quad n = 2, 4, \dots \tag{3.22}$$

$$E_0^{(2)} = \frac{1}{4}U_0 - \frac{1}{2}\omega(n - 1) \quad n = 1, 3, \dots \tag{3.23}$$

The size of gap Δ_E varies continuously along the curves from zero (at the starting point) to infinity:

$$\Delta_E^{(1)} = \frac{3}{2}U_0 + \omega(n + 1) \quad n = 2, 4, \dots \tag{3.24}$$

$$\Delta_E^{(2)} = -\frac{3}{4}U_0 - \frac{1}{2}\omega(n - 2) \quad n = 1, 3, \dots \tag{3.25}$$

(see figure 1). An example of such a potential is presented in figure 2.

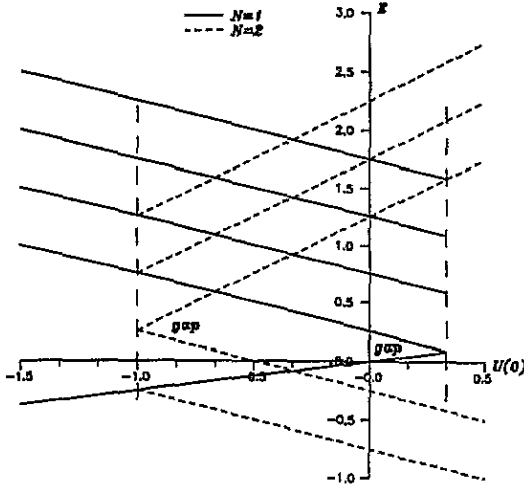


Figure 1. Change of energy levels along the curves (3.17) for the first two curves ($n = 1, 2$). As a parameter on the curves a value of $U_0 \equiv U(0)$ is chosen.

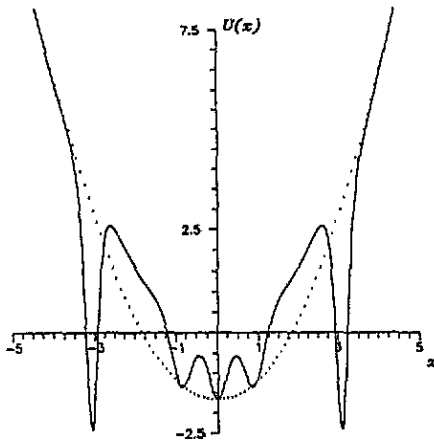


Figure 2. An example of a potential with a strictly equidistant spectrum except for a single gap. The lower part of the spectrum consists of four levels.

A detailed analysis of bifurcations of symmetric potentials in the quantum dynamical system under consideration is presented in [31].

The spectra that have the form of the equidistant continuation of the triplet of ground states are realized in potentials with asymptotics as given in the introduction. An example of this type of potential is shown in figure 3. If the parameters (C_1, C_2) vary in such a way

that the corresponding point (initially located inside one of the bounded regions) moves towards the bounding curve (3.17), then the distance between two (of three) 'ground' states of the spectrum decreases, so that at the curve two related equidistant sequences of levels merge and disappear (figure 4).

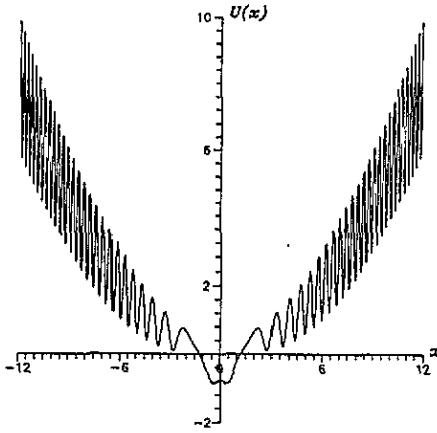


Figure 3. An example of a potential corresponding to the 'triplet' spectrum.

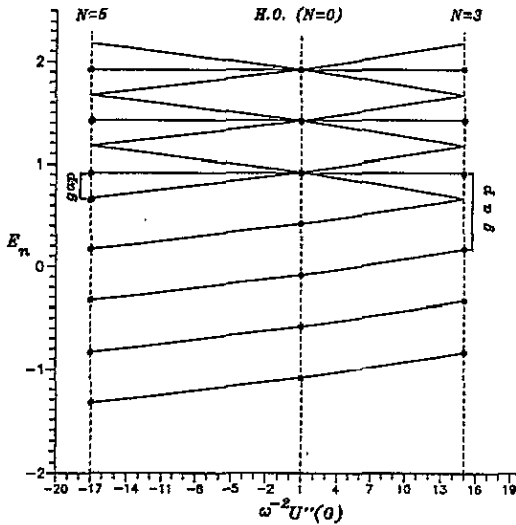


Figure 4. Transformation of energy levels by movement on the plane (C_1, C_2) from one curve ((3.17), $n = 3$, the right-hand broken vertical line) to the next curve ((3.17), $n = 5$, the left-hand broken vertical line) across the two bounded regions associated with the 'triplet' spectrum. These two regions make contact at the unique point (the middle vertical line) that belongs to the line $n = 0$ (the harmonic oscillator). In the 'natural' parameters this passage is equivalent to the segment $U(0) = -4/3, -17 < \omega^2 U_{xx}(0) < 15$. The full circles depict energy levels at the bounding curves.

Remark. Numerical simulations show that for non-symmetric potentials with different kinds of asymptotics at the left and right infinities the spectrum of H can be an infinite

equidistant continuation of two 'ground' states only.

A definition of operators N, \bar{N} in the form of polynomials of higher degree may lead to an increase in the number of arbitrarily located gaps in an equidistant spectrum and to spectra formed by equidistant continuation of multiplets of ground states.

Note that the problem of reconstruction of quantum dynamical systems (i.e. finding (H, L) -pairs of operators) in \mathbb{R}^1 for multivalued L -mappings requires the introduction of pseudo-spin degrees of freedom (extension of the class of dynamical systems). In the case when the sequence function leads to cycles of finite degree (and, correspondingly, to spectra containing finite number of levels) the problem may be required to pass on to shift operators L, L^\dagger that are matrices of finite range. The question of solvability of the problem remains open in the case that the sequence function has stable and unstable stationary points.

4. The classical analogue of the generalized nonlinear Fock approach

Consider the classical analogue of the method described above. For a classical Hamiltonian system with one degree of freedom let us introduce a complex function $L(x, p)$ defined by the relations

$$LL^* = N(H) \quad \{L, L^*\} = i\delta(H). \quad (4.1)$$

Here $N(H), \delta(H)$ are given real functions of H (H being the Hamiltonian function) and $\{.,.\}$ is a classical Poisson bracket. Note that in the quantum case this definition of the function L corresponds to the definition of the operator L by relations (2.2) (obviously an analogue of definition (2.1) cannot be used because $LL^* \equiv L^*L$).

By virtue of (4.1) we find that

$$\{H, L\} = -i \frac{\delta(H)}{\partial N / \partial H} L \equiv i\Omega(H)L. \quad (4.2)$$

In the action-angle variables (J, φ) equation (4.2) takes the form

$$\frac{\partial H}{\partial J} \frac{\partial L}{\partial \varphi} = i\Omega(H)L \quad (4.3)$$

and leads to the evident solution

$$L = \sqrt{N(H)} \exp\left(i \frac{\Omega(H)\varphi}{\partial H / \partial J}\right). \quad (4.4)$$

The function L defined by solution (4.4) is a function of the state of the Hamiltonian system if it is 2π -periodic in the angle variable φ and if $N(H) \geq 0$. Let these conditions be satisfied. Then

$$\frac{\partial H}{\partial J} = \Omega(H) \equiv -\frac{\delta(H)}{\partial N / \partial H} \quad N(H) \geq 0. \quad (4.5)$$

Taking into account that

$$J = \frac{1}{2}\pi \oint p \, dx$$

we write relation (4.5) in the form

$$\frac{\partial}{\partial H} \oint p \, dx = \frac{2\pi}{\Omega(H)} \equiv -\frac{2\pi(\partial N / \partial H)}{\delta(H)} \equiv T(H). \quad (4.6)$$

For the natural Hamiltonian system, $p = \sqrt{2(H - U(x))}$ and so relation (4.6) can be treated as the integral equation of the inverse problem of classical mechanics which determines the potential $U(x)$ from the given dependence of period T on energy [32].

Thus, equations (4.1) or (4.2) are directly related to the inverse problem of mechanics. One can treat the mapping $(x, p) \rightarrow (L, L^*)$ as a transition to a new representation where a state of the dynamical system on the complex plane L is determined by a rotating vector; its length and constant angular velocity are defined by the functions of energy N and δ .

5. The classical analogue of the description in the polynomial case

For dynamical systems leading to isochronous oscillations, $\Omega(H) = \omega \equiv \text{constant}$. In this case for the class of Hamiltonian functions $H = p^2/2 + U(x)$, equation (4.2) admits solutions in the class of polynomials of the form

$$L(x, p) = \sum_{m=0}^M L_m(x) p^m \quad M \geq 1. \tag{5.1}$$

Substitution of this expression into equation (4.2) generates an overdetermined system of linear equations in the functions $L_m(x)$, $m \leq M$. The condition of solvability for this system leads to a nonlinear equation for the potential $U(x)$. Its solutions contain the class of isochronous potentials.

As in the quantum case, we consider the problem of $M = 3$ and find that the solution of equation (4.2) is

$$L(x, p) = p^3 + i(\omega\xi)p^2 + [3V + (\omega\xi)^2]p + i[3V + (\omega\xi)^2] \left[\frac{1}{\omega} \frac{\partial V}{\partial \xi} + \omega\xi \right]. \tag{5.2}$$

Here $\xi \equiv (x - x_0)$, $V(\xi) = U(\xi) - A - (\omega\xi)^2/2$, $A = \text{constant}$, and the equation specifying a necessary class of potentials is

$$3 \frac{d}{d\xi} \left(V \frac{dV}{d\xi} \right) + (\omega\xi)^2 \frac{d^2V}{d\xi^2} + 3\omega^2\xi \frac{dV}{d\xi} = 0. \tag{5.3}$$

This equation possesses the integral

$$\frac{3}{2}V^2 - [3V + (\omega\xi)^2]\xi \frac{dV}{d\xi} = -C_1/2 \tag{5.4}$$

and has the two-parametric family of solutions

$$-4(\omega\xi)^2 = \frac{(3V^2 + C_1)^2}{V^3 + C_1V + C_2}. \tag{5.5}$$

Note that expressions (5.3)–(5.5) may also be obtained by formal transition ($\hbar \rightarrow 0$) to the classical limit from the corresponding quantum equations (3.10)–(3.12) though, in the general case, such a procedure should be used with precautions.

Simple calculations show that, in accordance with the original general relations (4.1),

$$LL^* \equiv N(H) = N_3H^3 + N_2H^2 + N_1H + N_0 \tag{5.6}$$

$$\{L, L^*\} \equiv \delta(H) = d_2H^2 + d_1H + d_0$$

where the structure parameters are determined by the expressions

$$\begin{aligned} \delta_2 &= 3\omega N_3 & \delta_1 &= 2\omega N_2 & \delta_0 &= \omega N_1 \\ N_3 &= 8 & N_2 &= -24A & N_1 &= -2C_1 + 24A^2 \\ N_0 &= -C_2 + 3C_1A + 8A^3. \end{aligned} \tag{5.7}$$

(Here A is the constant of potential renormalization in (5.2).)

We now describe a structure of potentials of the family (5.5). All the definite solutions (5.5) intersect the axis $\xi = 0$ at one of two points $U(0) = \pm U_0 + A$, where $U_0 = \sqrt{2C_1/3}$, and the value of C_2 (an analogue of the quantum second integral) is a parameter of the family. This family (5.5) is presented in figure 5. The condition $LL^* \equiv N(H) \geq 0$ is satisfied everywhere, and $N \equiv 0$ on the curves $U = U_0 + (\omega\xi)^2/2 + A$ and $U = (\omega\xi)^2/6 + A$.

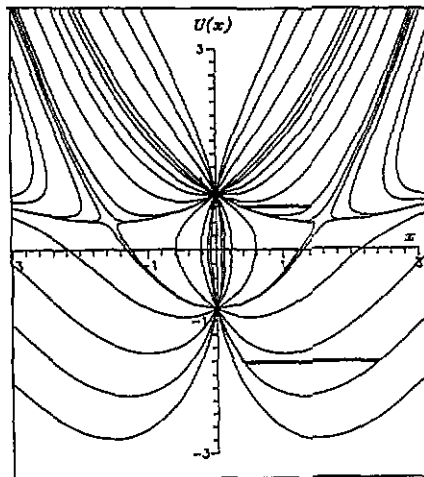


Figure 5. A family of potentials admitting isochronous motion in the classical case. The bold horizontal segments give examples of orbits in two different regions of isochronous motion.

Note that among this family there are only two potentials which are regular on the whole line—the harmonic oscillators $U = \pm U_0 + (\omega\xi)^2/2 + A$. In the limit case $C_1 = 0$ another harmonic oscillator exists: $U = (\omega\xi)^2/18$, which is an analogue of the same quantum solution (it corresponds to the zero value of the constant in the asymptotics presented in section 1). Other solutions have a break at the point $\xi = 0$. However, among the potentials that are defined on the half-line $\xi \geq 0$ we can single out groups of potentials admitting a finite motion in a certain range of energy. It is remarkable that this motion is strictly isochronous (i.e. its period does not depend on energy) in that range of energy values (see figure 5).

Acknowledgments

The authors would like to express their gratitude to Ja G Sinai for support of our investigations and useful discussions. We are also greatly indebted to the referees for useful bibliographical information.

Appendix A.

One can easily obtain equation (2.5) using a representation of N in the form

$$N(H) = \int d\tau N(\tau) \exp(iH\tau)$$

and taking into account that

$$[\exp(iH\tau), L] = \sum_{n=0}^{\infty} \frac{i\tau}{n!} \{(K + H)^n - LH^n\}.$$

Here

$$\begin{aligned} K^0 &\equiv L & K^1 &\equiv [H, L] & K^2 &\equiv [H, [H, L]], \dots \\ K^n &\equiv \underbrace{[H, [H, \dots [H, L] \dots]]}_n. \end{aligned}$$

By virtue of equations (2.4) the recurrent commutators K^n are determined by the relations $K^n = L\Omega^n(H)$. Hence,

$$[\exp(iH\tau), L] \Rightarrow L(\exp\{i(H + \Omega)\tau\} - \exp(iH\tau))$$

when using the representation of N in the integral form given above, and leads to the expressions

$$LD(H, \Omega) = 0 \quad D \equiv N(H + \Omega) - N(H) - \Delta(H)$$

and to the possibility of the extension of the definition of the characteristic operator $\Omega(H)$ by relation (2.5).

References

- [1] Shabat A B 1992 *Inverse Problems* **8** 303
- [2] Spiridonov V P 1992 *Phys. Rev. Lett.* **69** 398
- [3] Darboux J G 1882 *C. R. Acad. Sci., Paris* **94** 1456
- [4] Infeld I and Hull T E 1951 *Rev. Mod. Phys.* **23** 21
- [5] Green H S 1965 *Matrix Mechanics* (Groningen: Noordhoff)
- [6] Andrianov A A, Borisov N V and Ioffe M V 1984 *Teor. Mat. Fiz.* **61** 183
- [7] Witten E 1981 *Nucl. Phys.* **188** 513
- [8] Gendenshtein L E 1983 *Pis. Zh. Eksp. Teor. Fiz* **38** 299
- [9] Gendenshtein L E and Krive I V 1985 *Usp. Fiz. Nauk* **146** 553
- [10] Zheng W M 1983 *J. Phys. A: Math. Gen.* **16** 43
- [11] Fushchich W I and Nikitin A G 1987 *J. Phys. A: Math. Gen.* **20** 537
- [12] Beckers J, Deberg N and Nikitin A G 1991 *J. Phys. A: Math. Gen.* **24** L1269
- [13] Granovsky Ja I, Lutzenko I M and Zhedanov A S 1992 *Ann. Phys.* **217** 1
- [14] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
- [15] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
- [16] Hayashi T 1990 *Comm. Math. Phys.* **127** 129
- [17] Daskaloyannis C 1992 *J. Phys. A: Math. Gen.* **25** 2261
- [18] Daskaloyannis C and Ypsilantis K 1992 *J. Phys. A: Math. Gen.* **25** 4157
- [19] Harrel E 1980 *Comm. Math. Phys.* **75** 239
- [20] Davies E B 1984 *Ann. Phys.* **157** 166
- [21] Ashbaugh M S and Benguria R 1989 *Proc. Am. Math. Soc.* **105** 419
- [22] Yoffe A D 1993 *Adv. Phys.* **42** 173
- [23] Capasso F, Faist J, Sirtori C, Sivco P, Hutohinson A and Cho A 1994 *Proc. 22nd Int. Conf. Phys. Semicond. (Vancouver)*
- [24] Nieto M M and Truax D R 1993 *Phys. Rev. Lett.* **71** 2843
- [25] Fock V A 1932 *Z. Phys* **75** 622; 1932 *Z. Phys* **76** 952
- [26] Dubov S Yu, Eleonsky V M and Kulagin N E 1992 *Zh. Eksp. Teor. Fiz* **102** 814; 1992 *Sov. Phys.-JETP* **75** 446
- [27] Dubov S Yu, Eleonsky V M and Kulagin N E 1994 *Chaos* **4** 47
- [28] Eleonsky V M, Korolev V G and Kulagin N E 1994 *Chaos* **4** 583
- [29] Eleonsky V M, Korolev V G and Kulagin N E 1995 *Dokl. RAN* **342** 1-3
- [30] Veselov A P and Shabat A B 1993 *Funkz. Analiz Ego Pril.* **27** 1-21
- [31] Eleonsky V M and Korolev V G 1996 *Int. J. Bifurc. Chaos* **6** to appear
- [32] Landau L D and Lifshiz I M 1958 *Mechanics* (Moscow: Nauka)